Stat 201:
Introduction to Statistics

Standard 12: Probability – Basic Probability Results

CRACK BOOM

WHOA! WE SHOULD GET INSIDE!

IT'S OKAY! LIGHTNING ONLY KILLS ABOUT 45 AMERICANS A YEAR, SO THE CHANCES OF DYING ARE ONLY ONE IN 7,000,000. LET'S GO ON!
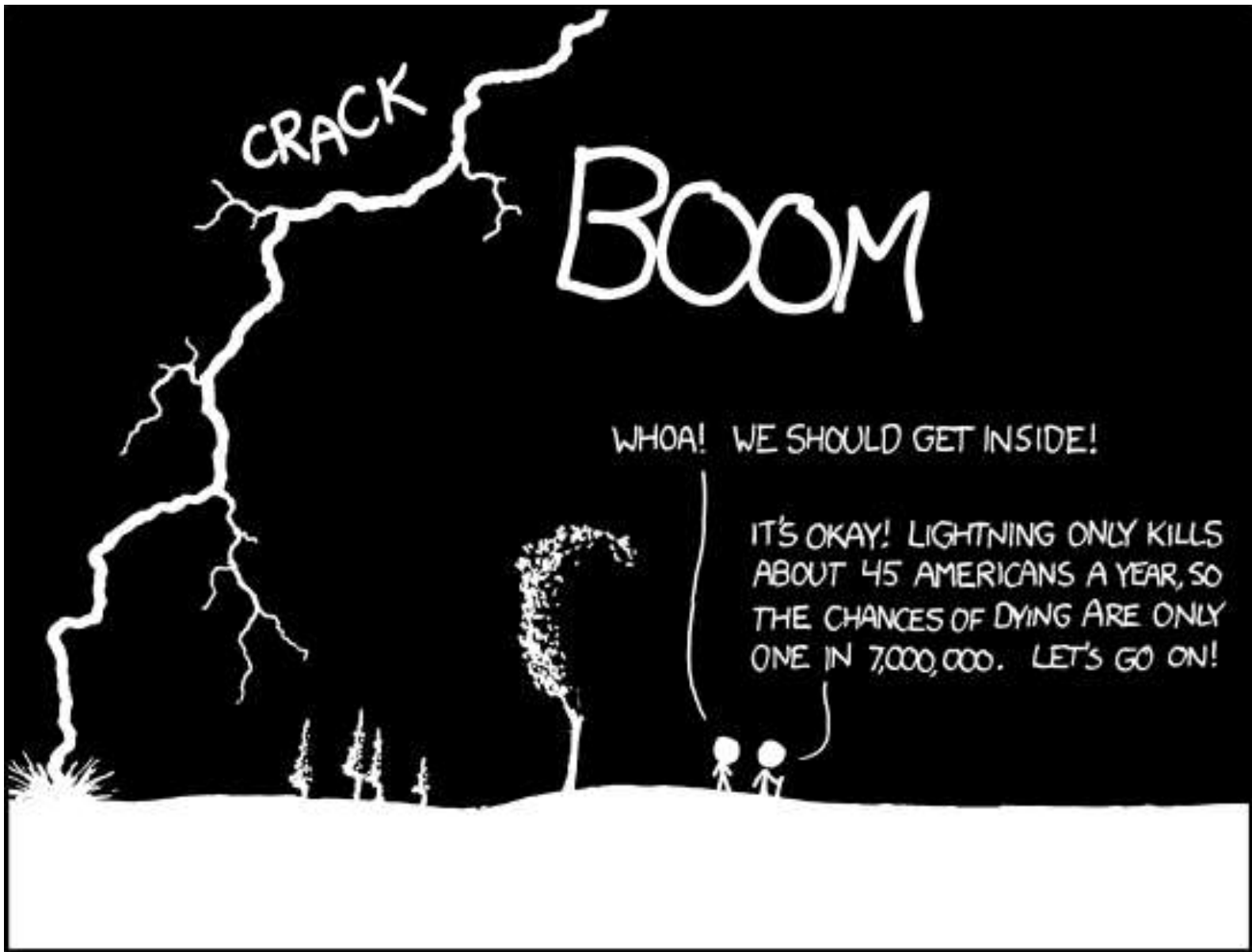
THE ANNUAL DEATH RATE AMONG PEOPLE WHO KNOW THAT STATISTIC IS ONE IN SIX.

Xkcd.com

Topics Up to This Point

- Chapter One
  - Study and experiment types and design
  - Sampling Methods and bias
- Chapter Two
  - Representing data – graphical and numerical
  - Measures of Central Tendency and Dispersion
  - Five number summary & Box plots
  - Outliers
- Chapter Three
  - Scatter Plots and Linear Regression

From *Naked Statistics: Probability*

- "Probability gives us tools for dealing with life's uncertainties. You shouldn't play the lottery. You should invest in the stock market if you have a long investment horizon (because stocks typically have the best long-term returns). You should buy insurance for some things, but not others. Probability can even help you maximize your winnings on game shows."

From *Naked Statistics: Probability*

- "Probability is the study of events and outcomes involving an element of uncertainty. Investing in the stock market involves uncertainty. So does flipping a coin, which may come up heads or tails. Flipping a coin four times in a row involves additional layers of uncertainty, because each of the four flips can result in a head or a tail."

From *Naked Statistics: Probability*

- "When it comes to risk, our fears do not always track with what the numbers tell us we should be afraid of. One of the striking findings from Freakonomics, by Steve Levitt and Stephen Dubner, was that swimming pools in the backyard are far more dangerous than guns in the closet. Levitt and Dubner calculate that a child under ten is one hundred times more likely to die in a swimming pool than from a gun accident."

From *Naked Statistics: Probability*

- "When it comes to risk, our fears do not always track with what the numbers tell us we should be afraid of. One of the striking findings from Freakonomics, by Steve Levitt and Stephen Dubner, was that swimming pools in the backyard are far more dangerous than guns in the closet. Levitt and Dubner calculate that a child under ten is one hundred times more likely to die in a swimming pool than from a gun accident."

Now… Probability

- This is very different but not unrelated
- This will help segue us into inferential statistics by allowing us to talk about the probability that an event will or will not happen and, more importantly, how likely it is to see a particular event given certain information

Introduction

- A lot of information really fast! Really, it's almost the whole chapter in 8 minutes!*
  - https://www.youtube.com/watch?v=YpvE0Co66nU

- Spinner probabilities
  - https://www.youtube.com/watch?v=QpfMwA0z_1Y

- No Punting*
  - https://www.youtube.com/watch?v=AGDaOJAYHfo

Probability - Definitions

- **Random Experiment –** Any action or observable experiment where the outcome is random

  - Possible outcomes are known but it is uncertain which will occur for any given observation.

  - **Examples:** Flipping a coin, rolling a die, etc

Probability - Definitions

- **Sample space –** the set of all possible outcomes in an experiment
  - **Examples**
    - **Rolling a die: S=**{1,2,3,4,5,6}
    - **Flipping a coin: S=**{Heads, Tails}

Probability - Definitions

- **Event** – We refer to a particular outcome occurring as an event
  - Examples: **A**=we rolled a 4, **A**= flipped heads

Randomness and Events

- Random Experiment: Flipping a Coin
  - Events: {Heads, Tails}

- Random Experiment: Rolling a die
  - Events: {1, 2, 3, 4, 5, 6}

- Random Experiment: The Weather
  - Events: {Sunny, Rainy, Snowy, Wintery Mix, Tornado, Earthquake, windy, hurricane, lightning storm}

Introduction to Probability

- **Probability –** The way we quantify uncertainty in random experiments

- **Randomness –** Possible outcomes are known but it is uncertain which will occur for any given observation.
  - **Note:** Random phenomena is highly variable so we look at patterns in the long run

Probability

- Think about a football game. The captains of both teams go to the center of the field to call heads or tails in hopes that they'll be able to choose who gets the ball first.

- **Q:** What is the probability that the home team wins the toss if they call heads?
  - Clearly, we should all answer they have a 50/50 shot – 50% chance – probability of 0.5.

Probability – How to Calculate

- We used this calculation before, but let's define it here:

- *Probability of A*:

$$P(A) = \frac{N(A)}{N(S)} = \frac{(Number\ of\ ways\ A\ can\ happen)}{(Total\ number\ of\ outcomes)}$$

  – Probability Properties:
    - Probability cannot be negative
    - $0 \leq P(A) \leq 1$
    - The sum of the probability of all possible outcomes of a random experiment must equal one

Probability – How to Calculate

- If we can't enumerate the number of ways we can use observations and their proportions to approximate the probability – **Empirical Probability**

- $\widehat{P(A)} = \dfrac{N(A)}{N(S)} = \dfrac{(Number\ of\ times\ event\ A\ happened)}{(Total\ number\ of\ trials)}$

  – Probability Properties:

    - Probability cannot be negative
    - $0 \leq \widehat{P(A)} \leq 1$
    - The sum of the probability of all possible outcomes of a random experiment must equal one

Probability – How to Calculate

- **Example: Rolling a die**
  - **Sample Space =** S = {1,2,3,4,5,6}
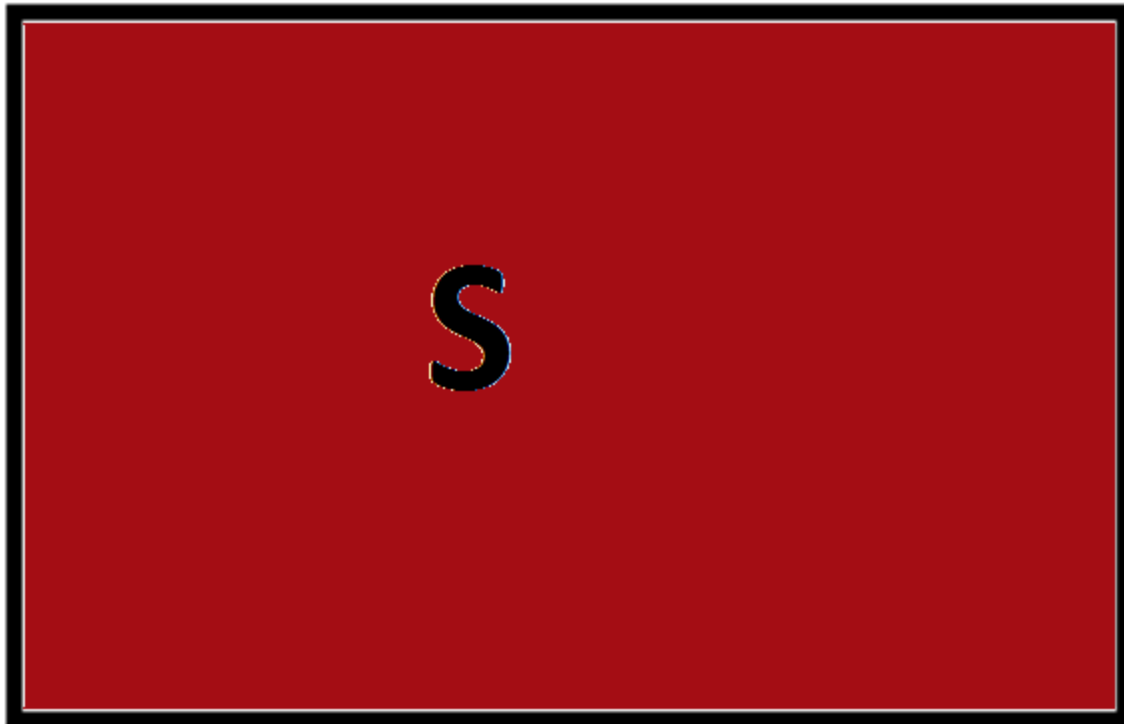 - Let **event A =** roll a multiple of three = {3,6}
  - **Probability** we roll a multiple of three:
  $$P(A = \{3,6\}) = \frac{N(A)}{N(S)} =$$
  $$\frac{(Number\ of\ ways\ we\ can\ roll\ a\ 3\ or\ 6)}{(Total\ number\ of\ outcomes)} = \frac{2}{6} = \frac{1}{3}$$

Adjectives for Events: Complement

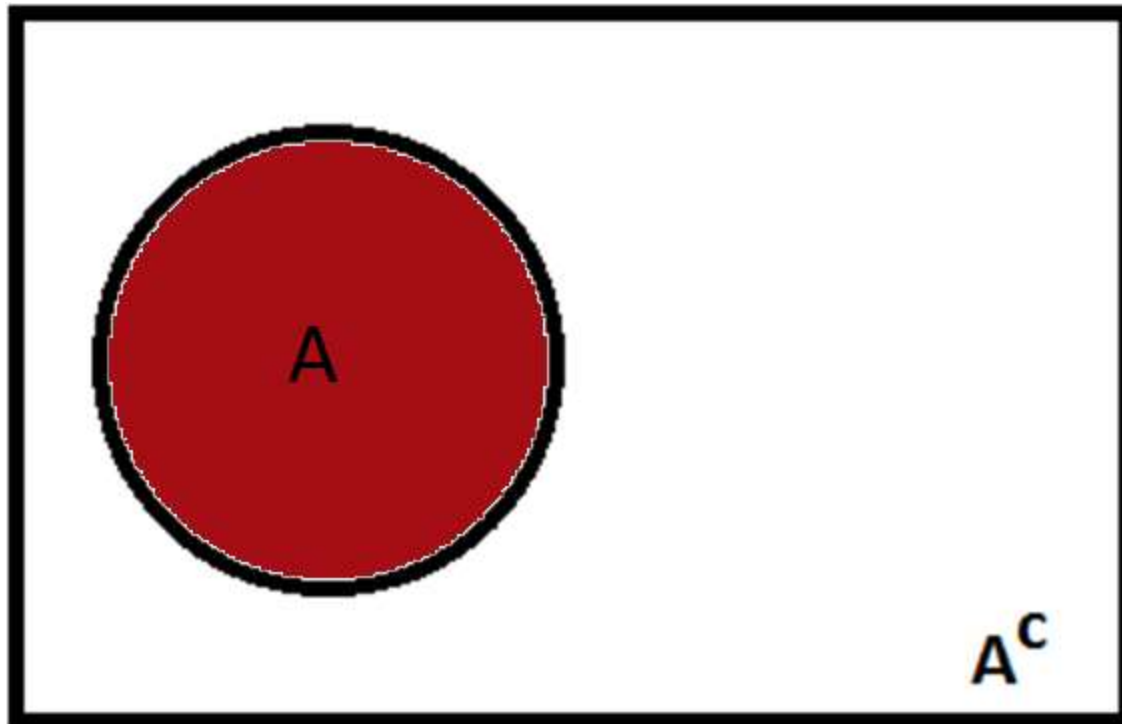- **The Sample Space** is the collection of all outcomes of a random experiment

Adjectives for Events: Complement

- **The Sample Space** = S = {1,2,3,4,5,6}
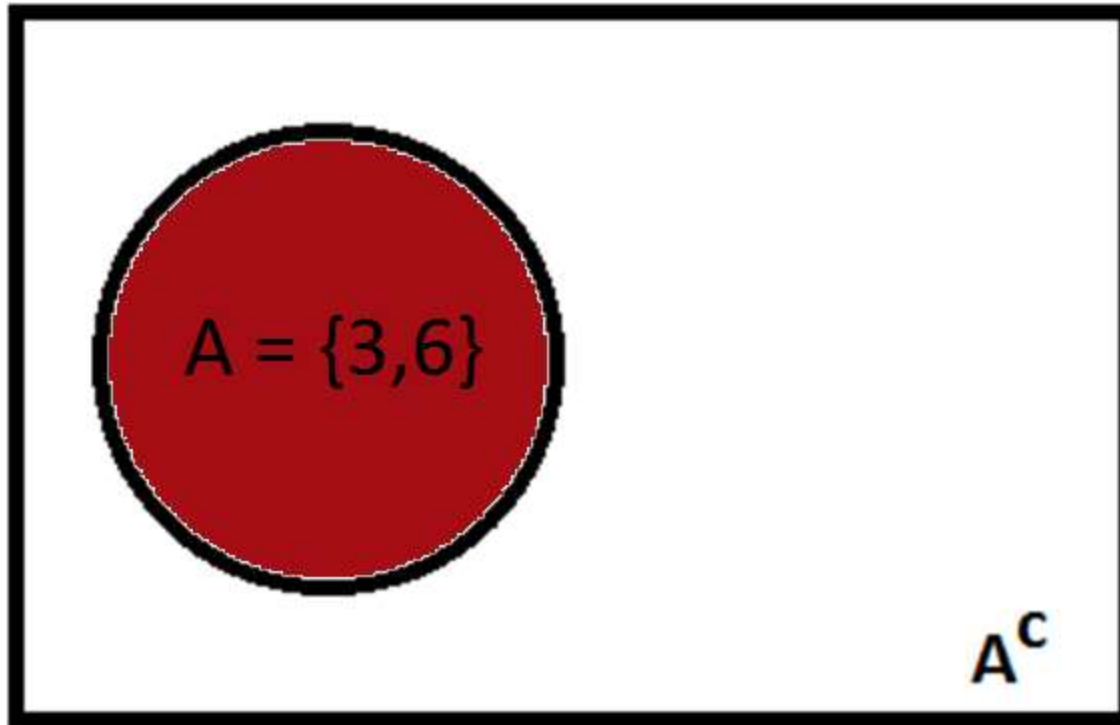
$$S = \{1,2,3,4,5,6\}$$

Adjectives for Events: Complement

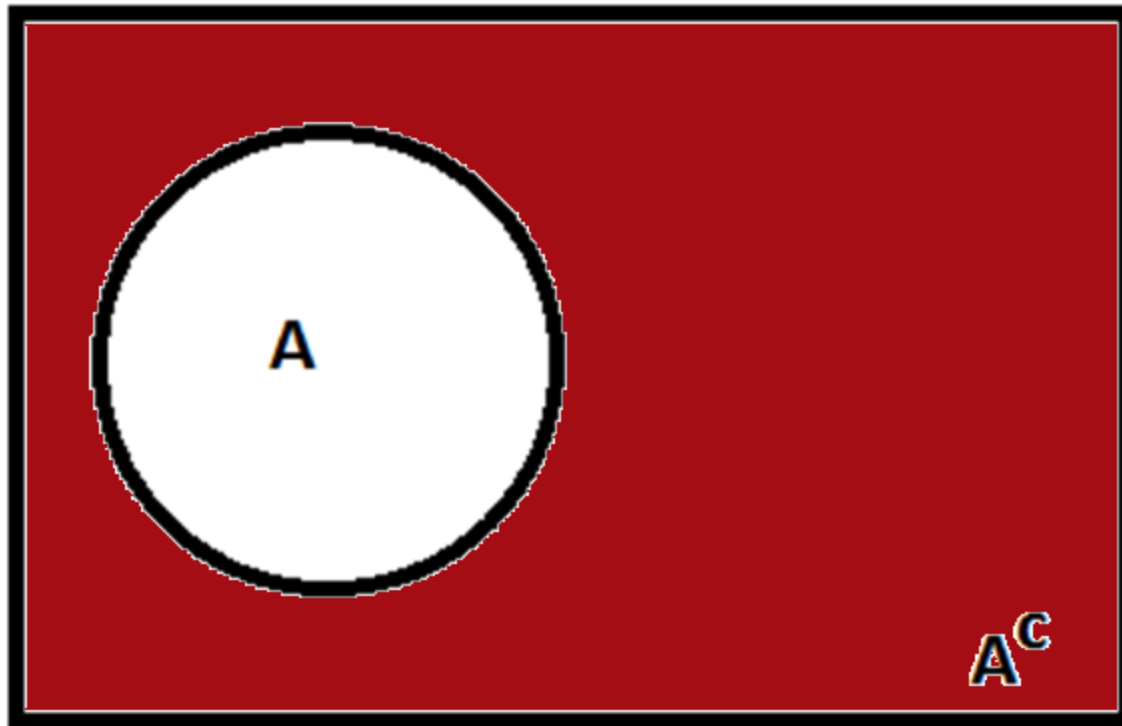- **Event A** is the collection of all outcomes that belong to the event A

Adjectives for Events: Complement
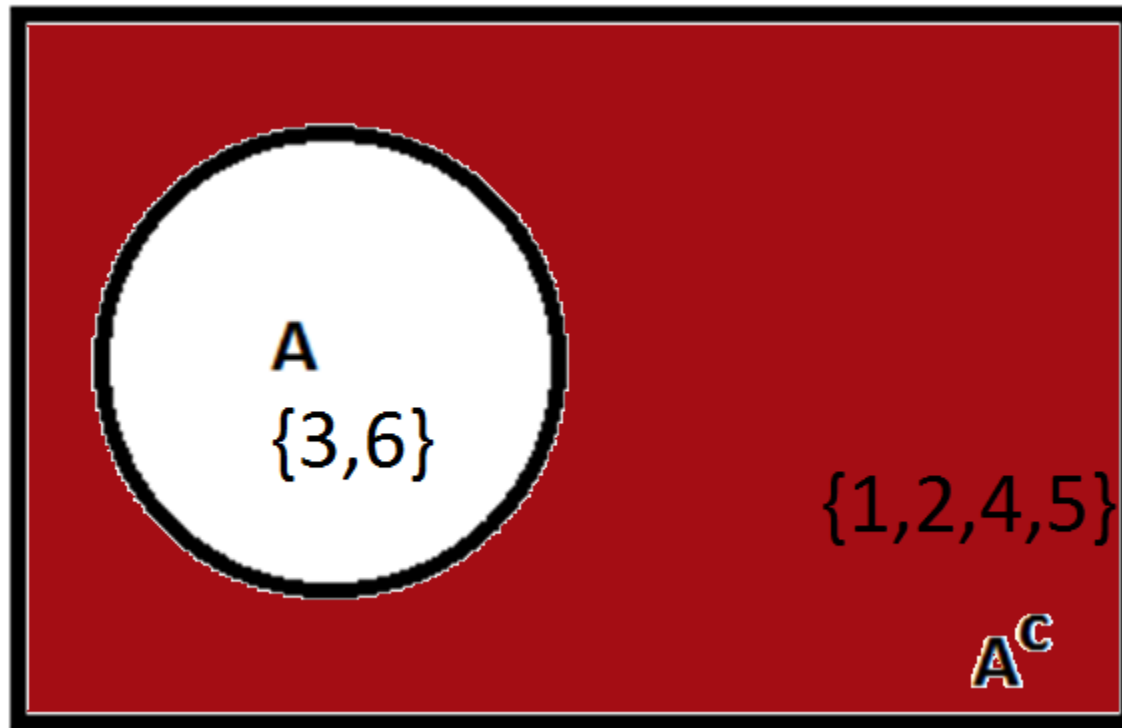
- **Event A** = {3,6}

Adjectives for Events: Complement

- **The complement** of an event A is the set of all outcomes not in A
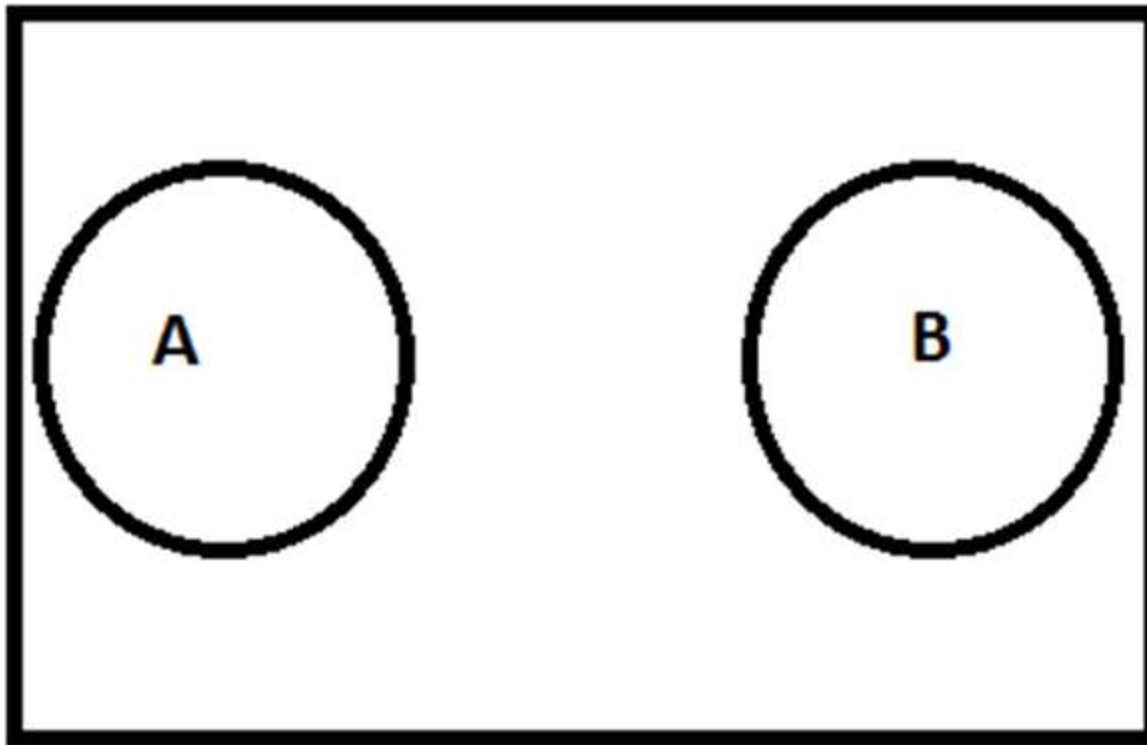
Adjectives for Events: Complement

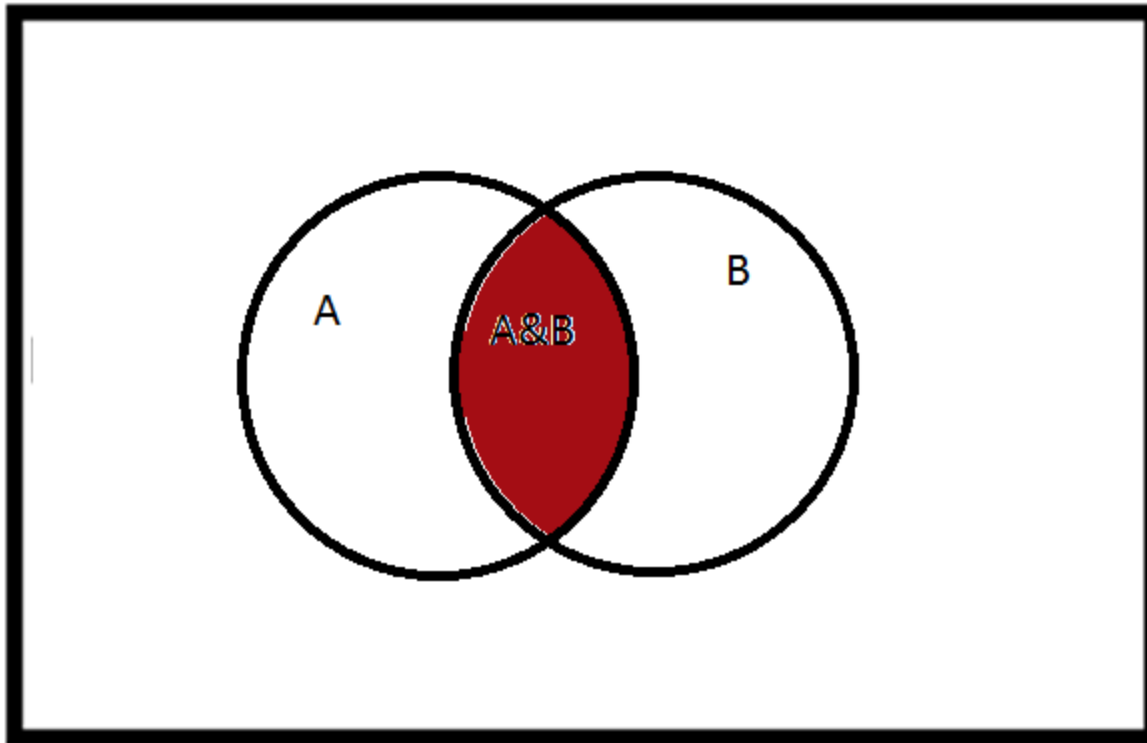- **The complement** of $A = A^c = \{1,2,4,5\}$

Adjectives for Events: Mutually Exclusive (Disjoint)

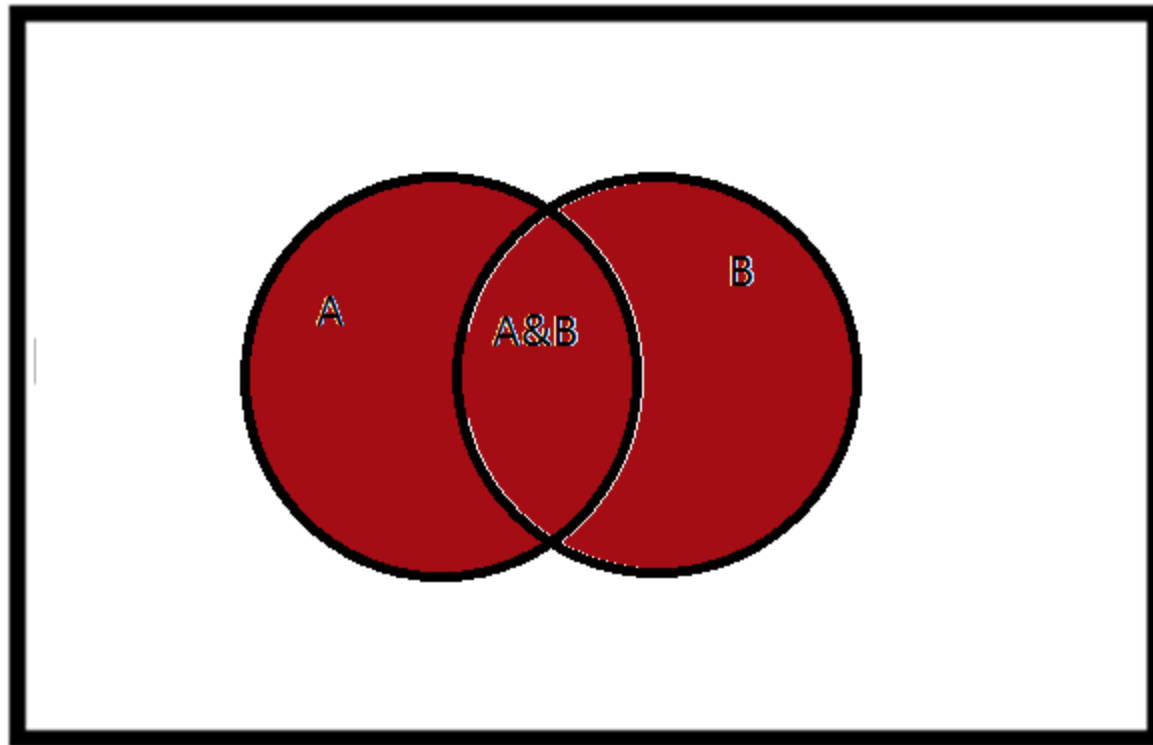- Two events are **mutually exclusive (disjoint)** if they do not have any common outcomes

Adjectives for Events: Intersection

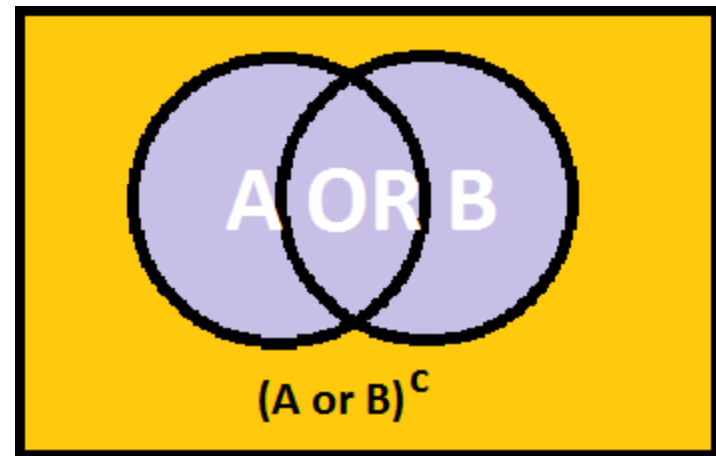- The **intersection** of A and B $(A \cap B)$ consists of outcomes that are both in A and B

Adjectives for Events: Union

- The **union** of A and B $(A \cup B)$ consists of outcomes that are A or B or both

Putting it Together: Summary
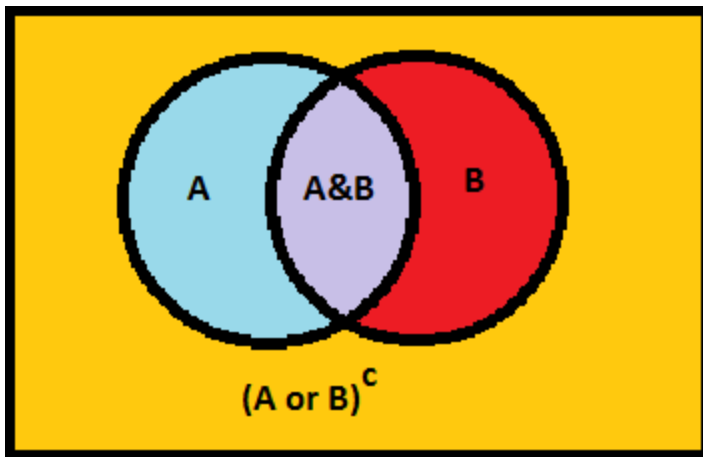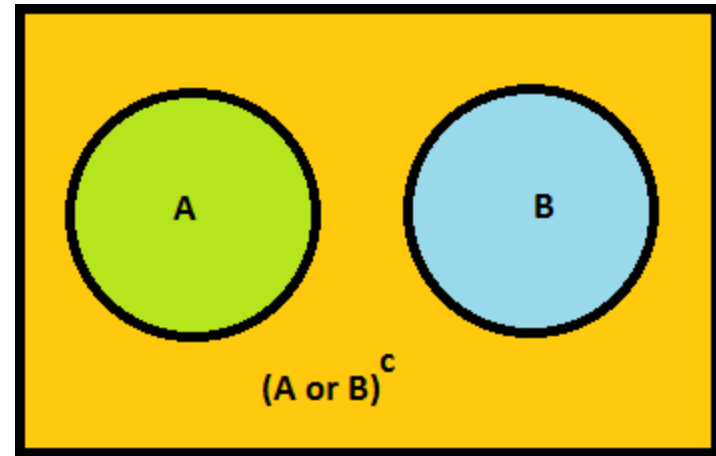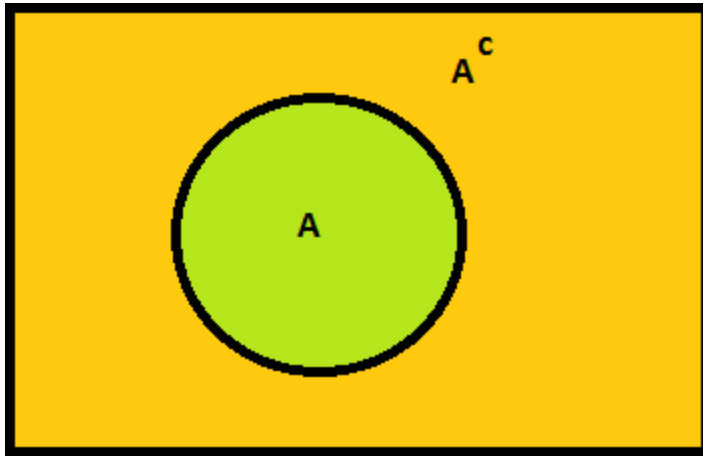
Probability Rules

- **Complement Rule:**
  - The probability of something not happening is 1 minus the probability of it happening
  $$P(A^c) = P(S) - P(A)$$
  $$P(A^c) = 1 - P(A)$$
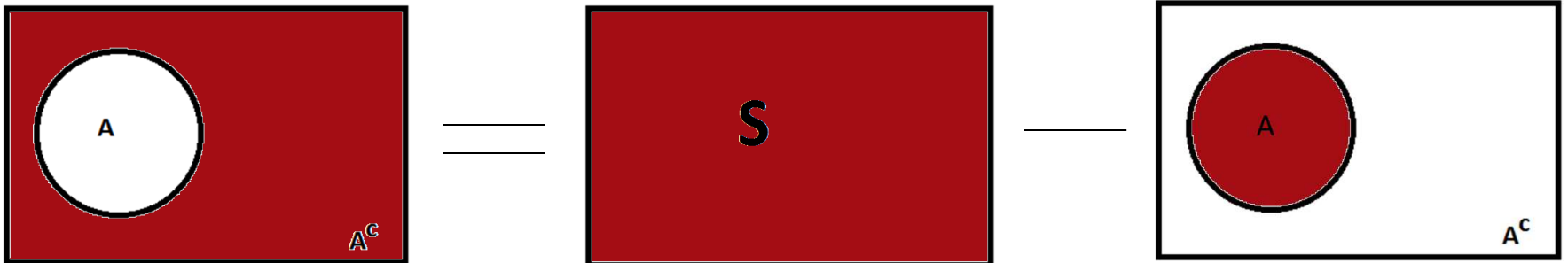
Probability Rules

- **Conditional Probabilities**
  - The probability of A given B:

$$P(A|B) = \frac{P(A \ and \ B)}{P(B)} = \frac{P(A \cap B)}{P(B)} =$$
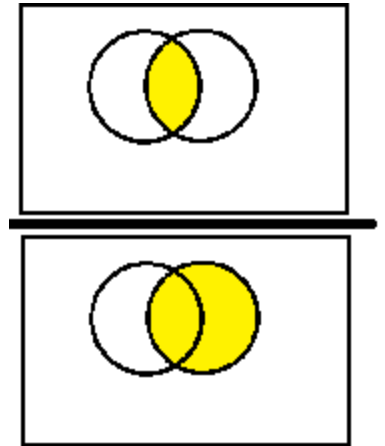
Probability:
Additive Rule of Probability

- $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

  – The probability of A or B happening is the probability of A, plus the probability of B, minus the probability of A & B because we're double counting the probability that they both happen

  – **For mutually exclusive or disjoint events $P(A \text{ or } B) = P(A) + P(B)$**
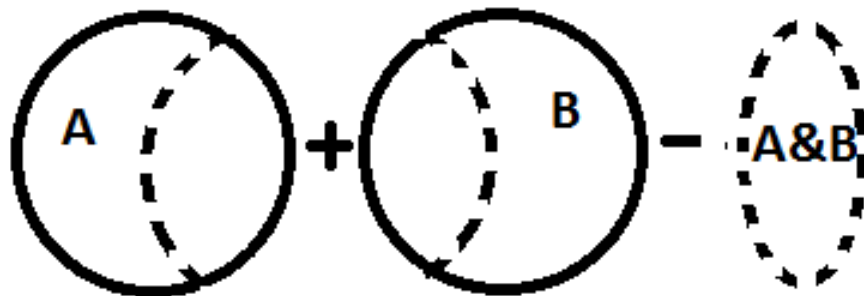
Probability: Additive Rule of Probability for Mutually Exclusive Events

- For mutually exclusive A and B we know that
  $P(A \ and \ B)$=$P(A \ \cap B) = 0$

- Therefore the additive rule becomes:
- $P(A \ or \ B) = P(A) + P(B)$
- $P(A \ \cup B) = P(A) + P(B)$

Probability Rules

- **Remember, in math or means one, the other or both.**
- <u>**WITHOUT MATH:**</u> "You can have soup or salad."
  - You could only order one
- <u>**WITH MATH:**</u> "You can have soup or salad."
  - You could order soup, salad or both soup and salad

Probability Rules

- The probability of A and B happening is either:
  - The probability of A times the probability of B given A
  - The probability of B times the probability of A given B

- $\mathrm{P}(A \text{ and } B) = \mathrm{P}(A) * \mathrm{P}(B|A)$
$$= \mathrm{P}(B) * \mathrm{P}(A|B)$$
- $\mathrm{P}(A \cap B) = \mathrm{P}(A) * \mathrm{P}(B|A)$
$$= \mathrm{P}(B) * \mathrm{P}(A|B)$$

Law of Total Probability

- $P(A) =$
$P(A \cap B_1) + P(A \cap B_2) + \cdots + P(A \cap B_k)$
$\qquad = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \cdots$
$\qquad\qquad + P(A|B_k)P(B_k)$

Where $B_1 \cup B_2 \cup \ldots \cup B_k = S = sample\ space$

Law of Total Probability
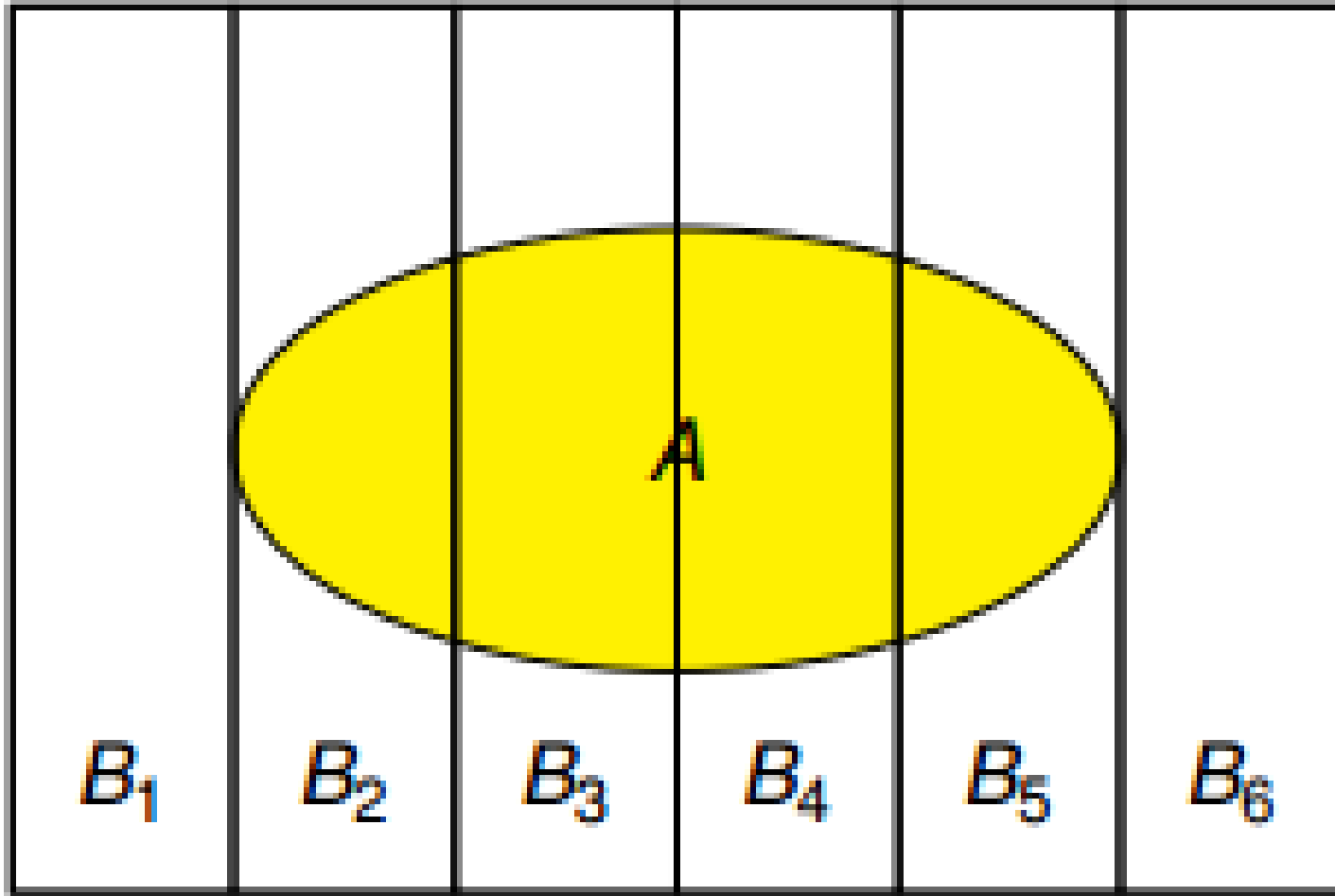
Law of Total Probability

- In this case $B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5 \cup B_6 = S = sample\ space$
- So we can find P(A) by looking at it in pieces:

$$P(A) = \mathrm{P}(A \cap B_1) + \mathrm{P}(A \cap B_2) + \mathrm{P}(A \cap B_3)$$
$$+\mathrm{P}(A \cap B_4) + \mathrm{P}(A \cap B_5) + \mathrm{P}(A \cap B_6)$$

$$= \mathrm{P}(A|B_1)\mathrm{P}(B_1) + \mathrm{P}(A|B_2)\mathrm{P}(B_2)+\mathrm{P}(A|B_3)\mathrm{P}(B_3)$$
$$+\mathrm{P}(A|B_4)\mathrm{P}(B_4) + \mathrm{P}(A|B_5)\mathrm{P}(B_5)+\mathrm{P}(A|B_6)\mathrm{P}(B_6)$$
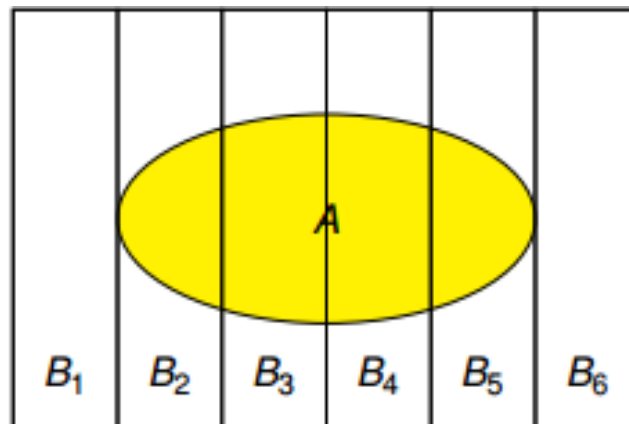
Law of Total Probability

- Consider a shuffled deck of cards? What is the probability the second card is a King?

- **First** we condition whether the first card is a king or not – think "sequentially"

- If K_1 = the first card is a king and K_2 = the second card is a king then:
$$P(K_2) = P(K_2|K_1)\mathrm{P(K_1)} + \mathrm{P(K_2|K_1^c)P(K_1^c)}$$

- Note: $K_1 \cup K_1^C = all\ possible\ first\ cards$

At Least Probabilities

- Here, we'll learn by example: How can we find the probability that we roll at least a 5 when using a six sided die.
  - X, a random variable, is the number rolled
  - Well, what is at least 5?
    - 5 and greater: 5 and 6
    - A= {5,6}

At Least Probabilities

- **The probability that we roll at least a five:**

$$P(A = \{5,6\}) = P(X \geq 5) = P(X = 5) + P(X = 6)$$

$$= \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \mathbf{\frac{1}{3}}$$

- **With Complement Rule:**

$$P(A = \{5,6\}) = P(X \geq 5) = 1 - P(X < 5)$$

$$= 1 - \big(P(X = 4) + P(X = 3) + P(X = 2) + P(X = 1)\big)$$

$$= 1 - \left[\left(\frac{1}{6}\right) + \left(\frac{1}{6}\right) + \left(\frac{1}{6}\right) + \left(\frac{1}{6}\right)\right] = 1 - \frac{4}{6} = \frac{6}{6} - \frac{4}{6} = \frac{2}{6} = \mathbf{\frac{1}{3}}$$

Probabilities from Tables

- We cannot survey the population, it's far too big.

- Instead, we look at large surveys or experiments and we invoke the **Law of Large Numbers** to get what we refer to as **empirical probabilities**

- Here – we talk about our participants as the sample space and calculate probabilities only using them.

Example: Probability

↓Velociraptor safe windows?	Survived Velociraptor Attacks (S)	Devoured by Velociraptors (D)	Total
Yes (Y)	412,368	510	**412,878**
No (N)	16,001	162,527	178,528
Total	428,369	163,037	**591,406**

- The probability a randomly selected participant had Velociraptor safe windows:

$$\widehat{P(Y)} = \frac{number\ of\ Y\ observations}{total\ number\ of\ observations} =$$

$$\frac{412,878}{591,406} = .69812954$$

Example: Probability

↓Velociraptor safe windows?	Survived Velociraptor Attacks (S)	Devoured by Velociraptors (D)	Total
Yes (Y)	412,368	510	412,878
No (N)	16,001	162,527	178,528
Total	**428,369**	163,037	**591,406**

- The probability a randomly selected participant survived the Velociraptor attacks

- $\widehat{P(S)} = \dfrac{number\ of\ S\ observations}{total\ number\ of\ observations} =$

$\dfrac{428,369}{591,406} = .72432305$

Example: Probability

↓Velociraptor safe windows?	Survived Velociraptor Attacks (S)	Devoured by Velociraptors (D)	Total
Yes (Y)	412,368	510	412,878
No (N)	16,001	162,527	178,528
Total	428,369	**163,037**	**591,406**

- The probability a randomly selected participant was devoured

- $\widehat{P(D)} = \dfrac{number\ of\ D\ observations}{total\ number\ of\ observations} = \dfrac{163,037}{591,406} = .27567695$

Example: Probability

- The probability a randomly selected participant did not survive the Velociraptor attacks: (Same logic as the previous slide)

- We know $P(S) = .72432305$

- $\widehat{P(S^c)} = 1 - P(S) = 1 - .72432305$
$$= .27567695 = P(D)$$

Example: Probability

↓Velociraptor safe windows?	Survived Velociraptor Attacks (S)	Devoured by Velociraptors (D)	Total
Yes (Y)	**412,368**	510	**412,878**
No (N)	16,001	162,527	178,528
Total	**428,369**	163,037	**591,406**

- The probability a randomly selected participant had Velociraptor safe windows **or** survived the attack:

- $P(\widehat{S \cup Y}) = \dfrac{number\ of\ S\ or\ Y\ observations}{total\ number\ of\ observations} =$
$$\dfrac{428{,}369+412{,}878-412{,}368}{591{,}406} = .72518541$$

Example: Probability

↓Velociraptor safe windows?	Survived Velociraptor Attacks (S)	Devoured by Velociraptors (D)	Total
Yes (Y)	**412,368**	510	412,878
No (N)	16,001	162,527	178,528
Total	428,369	163,037	**591,406**

- The probability a randomly selected participant survived the Velociraptor attacks **and** had Velociraptor safe windows:

- $P(\widehat{S \cap Y}) = \dfrac{number\ of\ S\&Y\ observations}{total\ number\ of\ observations}$

$$= \frac{412,368}{591,406} = .69726719$$

Example: Probability

↓Velociraptor safe windows?	Survived Velociraptor Attacks (S)	Devoured by Velociraptors (D)	Total
Yes (Y)	412,368	510	412,878
No (N)	16,001	162,527	178,528
Total	428,369	163,037	591,406

- The probability a randomly selected participant had Velociraptor safe windows **or** survived the attack:
- **Let's try the formula:**
- $P(\widehat{S \cup Y}) = P(S) + P(Y) - P(S \cap Y)$

$$= .72432305 + .69812954 - .69726719$$
$$= .7251854$$

Example: Probability

↓Velociraptor safe windows?	Survived Velociraptor Attacks (S)	Devoured by Velociraptors (D)	Total
Yes (Y)	412,368	510	412,878
No (N)	16,001	162,527	178,528
Total	428,369	163,037	591,406

- Now, let's think about a word problem.
- Let's say Velociraptors came back, maybe there was a cryogenically frozen one that thawed out. Anyway, **would it be worth getting the Velociraptor safe windows?**

Example: Probability

↓Velociraptor safe windows?	Survived Velociraptor Attacks (S)	Devoured by Velociraptors (D)	Total
Yes (Y)	412,368	510	412,878
No (N)	16,001	162,527	178,528
Total	428,369	163,037	591,406

- At first look…
  - We see that a lot more people survived the Velociraptor attacks with the Velociraptor safe windows
  - We see that a lot more were devoured without the Velociraptor safe windows
- **But** counts can be misleading because there are far more people with the windows than without

Summary!

Probability

Type	Description
Random Experiment	Any action or observable experiment where the outcome is random; possible outcomes are known but it is uncertain which will occur for any given observation.
Sample space	The set of all possible outcomes in an experiment
Event	We refer to a particular outcome occurring as an event
Probability of event A $0 \leq P(A) \leq 1$	$$P(A) = \frac{N(A)}{N(S)} = \frac{(Number\ of\ ways\ A\ can\ happen)}{(Total\ number\ of\ outcomes)}$$

Probability

Type	Description
Probability of event A $0 \leq P(A) \leq 1$	$$P(A) = \frac{N(A)}{N(S)} = \frac{(Number\ of\ ways\ A\ can\ happen)}{(Total\ number\ of\ outcomes)}$$
Complement Rule ('not A')	$$P(A^c) = 1 - P(A)$$
Or Probability ('A or B')	$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$
And Probability ('A and B')	$$P\left(A \bigcap B\right) = P(A) * P(B|A)$$ $$= P(B) * P(A|B)$$

Probability

Type	Description
Intersection	The **intersection** of A and B consists of outcomes that are both in A and N
Union	The **union** of A and B consists of outcomes that are A or B or both

Other Types of Events

- **Impossible:** when the probability of the event occurring is zero

- **Certain:** when the probability of the event occurring is one

- **Unusual:** when the probability of the event occurring is low. We consider low to be less than 0.05.